Lab 10 solutions

1. Let $K = v$ consist of a single vertex.
   (a) Compute the chain groups C_0 and C_1.
   (b) Compute the homology groups $H_0(K)$ and $H_1(K)$.

   Solution:

   (a) The only simplex is a 0-simplex:

   $$C_0 = \langle v \rangle \cong Z, \quad C_1 = 0.$$

   (b)
   $$H_0(K) = \ker(d_0)/\mathrm{im}(d_1).$$

   Since $d_0 = 0$, we have $\ker(d_0) = C_0 = Z$ and $\mathrm{im}(d_1) = 0$. Thus:

   $$H_0(K) \cong Z, \quad H_1(K) = 0.$$

2. Let K consist of two vertices v_0, v_1 and one edge $[v_0v_1]$.
   (a) Compute the boundary map ∂_1.
   (b) Compute the homology groups $H_0(K)$ and $H_1(K)$.

   Solution:

   (a)
   $$\partial_1([v_0v_1]) = v_1 - v_0.$$

   (b)
   $$C_0 \cong Z^2, \quad C_1 \cong Z.$$

   The image of d_1 is:
   $$\mathrm{im}(d_1) = \langle v_1 - v_0 \rangle.$$

   Thus:
   $$H_0(K) = Z^2/\langle v_1 - v_0 \rangle \cong Z.$$

   Also $\ker(d_1) = 0$, so:
   $$H_1(K) = 0.$$

3. Let K be the simplicial complex consisting of two disjoint edges.
   (a) Compute $H_0(K)$.
   (b) Interpret your answer in terms of connected components.

   Solution:

   (a) There are two connected components, so:

   $$H_0(K) \cong Z^2.$$

   (b) Each connected component contributes one generator to H_0. Thus H_0 counts connected components.

4. Let K be the boundary of a tetrahedron (all triangular faces included, but not the 3-simplex).
   (a) Describe the chain groups C_0, C_1, C_2.
   (b) Argue that $H_2(K) \cong Z$.
   (c) Argue that $H_1(K) = 0$.

   Solution:

   (a) A tetrahedron has:

   $$C_0 \cong Z^4, \quad C_1 \cong Z^6, \quad C_2 \cong Z^4.$$

   (b) The sum of all four triangular faces forms a 2-cycle. Since there is no 3-simplex, it is not a boundary. Hence:

   $$H_2(K) \cong Z.$$

   (c) Every 1-cycle is the boundary of a combination of triangles. Thus:

   $$\ker(d_1) = \operatorname{im}(d_2) \quad \Rightarrow \quad H_1(K) = 0.$$

5. Let K contain a 2-simplex $[v_0v_1v_2]$.
   (a) Compute $\partial_2([v_0v_1v_2])$.
   (b) Explain how to recognize from a picture that a 1-cycle is a boundary.

   Solution:

   (a)

   $$\partial_2([v_0v_1v_2]) = [v_1v_2] - [v_0v_2] + [v_0v_1].$$

   (b) A 1-cycle is a boundary if it is the boundary of a 2-simplex. Geometrically, this means the loop encloses a filled triangle.

6. Let K be any simplicial complex.

   (a) Show that if two vertices are connected by an edge, then they represent the same class in $H_0(K)$.

   (b) Conclude that $H_0(K)$ depends only on the connected components of K.

Solution:

(a) If v_0 and v_1 are connected by an edge:

$$\partial_1([v_0 v_1]) = v_1 - v_0.$$

Thus $v_1 - v_0 \in \text{im}(d_1)$, so v_0 and v_1 represent the same class in H_0.

(b)

$$H_0(K) \cong Z^{\#\text{connected components}}.$$

7. Let $c = \sum a_i[v_i v_{i+1}]$ be a 1-chain.

   (a) Write out the condition $\partial_1(c) = 0$.

   (b) Interpret this condition as a statement about coefficients at each vertex.

   (c) Explain why this corresponds to having "no endpoints."

Solution:

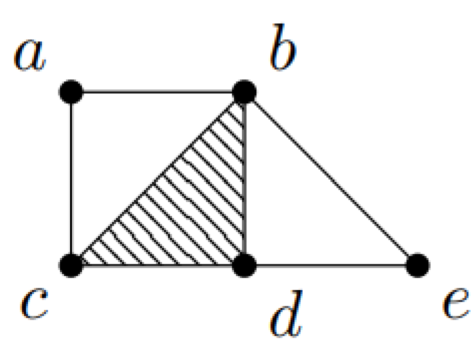
(a)

$$\partial_1(c) = \sum a_i(v_{i+1} - v_i).$$

(b) At each vertex, the sum of coefficients of incoming edges equals the sum of outgoing edges.

(c) This means there are no endpoints: every vertex is balanced. Hence c is a cycle (a closed loop).

8. Compute the simplicial homology of the following simplicial complex:



Solution:

The simplicial complex K has:

- Vertices:
$$\{a, b, c, d, e\}$$
- Edges:
$$[ab], [ac], [cd], [bd], [be], [de]$$
- One 2-simplex:
$$[cbd]$$

The chain groups are:

$$C_0(K) \cong Z^5, \quad C_1(K) \cong Z^6, \quad C_2(K) \cong Z.$$

Since all vertices are connected through edges, the simplicial complex has one connected component. Therefore:

$$H_0(K) \cong Z.$$

For $H_1(K)$, we analyze cycles and boundaries.

A natural 1-cycle in the complex is the loop:

$$[ab] + [bd] + [dc] + [ca].$$

Now consider the 2-simplex $[cbd]$. Its boundary is:

$$\partial_2([cbd]) = [bd] - [cd] + [cb].$$

Rewriting this, we obtain:

$$[bd] + [dc] = [bc].$$

Substituting into the loop:

$$[ab] + [bd] + [dc] + [ca] = [ab] + [bc] + [ca].$$

Thus the original loop is homologous to the triangle $[abc]$. However, this triangle is *not* a 2-simplex in the complex, so we must check whether this produces a nontrivial homology class.

Observe that every 1-cycle can be reduced using the relation coming from $\partial_2([cbd])$. In fact, all cycles are generated by boundaries of 2-simplices, and no independent 1-cycle remains.

Hence:

$$\ker(d_1) = \operatorname{im}(d_2),$$

and therefore:

$$H_1(K) = 0.$$

$$H_2(K) = \ker(d_2)/\operatorname{im}(d_3).$$

There is only one 2-simplex $[cbd]$, and:

$$\partial_2([cbd]) \neq 0.$$

Thus:

$$\ker(d_2) = 0.$$

Since $C_3 = 0$, we have $\operatorname{im}(d_3) = 0$, so:

$$H_2(K) = 0.$$

Final result

$$H_0(K) \cong Z, \quad H_1(K) = 0, \quad H_2(K) = 0.$$

Interpretation

- H_0 detects one connected component.
- $H_1 = 0$ means there are no 1-dimensional holes (all loops are filled or reducible).
- $H_2 = 0$ means there are no 2-dimensional voids.